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Abstract

We give the definition and explore the algebraic structure of a class of quantum symmetries, called topological symmetries, which are generalizations of supersymmetry in the sense that they involve topological invariants similar to the Witten index. A topological symmetry (TS) is specified by an integer $n > 1$, which determines its grading properties, and an n-tuple of positive integers $(m_1, m_2, \cdots, m_n)$. We identify the algebras of supersymmetry, $p = 2$ parasupersymmetry, and fractional supersymmetry of order n with those of the $\mathbb{Z}_2$-graded TS of type $(1, 1)$, $\mathbb{Z}_2$-graded TS of type $(2, 1)$, and $\mathbb{Z}_n$-graded TS of type $(1, 1, \cdots, 1)$, respectively. We also comment on the mathematical interpretation of the topological invariants associated with the $\mathbb{Z}_n$-graded TS of type $(1, 1, \cdots, 1)$. For $n = 2$, the invariant is the Witten index which can be identified with the analytic index of a Fredholm operator. For $n > 2$, there are n independent integer-valued invariants. These can be related to differences of the dimension of the kernels of various products of n operators satisfying certain conditions.

1 Introduction

Supersymmetric quantum mechanics was originally introduced as a toy model used to study some of the features of supersymmetric field theories [1]. This simple toy model

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has, however, proven to be a very useful tool in dealing with a variety of problems in quantum and statistical mechanics [2, 3, 4]. Supersymmetric quantum mechanics has also been used to derive some of the very basic results of differential topology. Among these are the supersymmetric derivation of the Morse inequalities [5] and supersymmetric proofs of the Atiyah-Singer index theorem [6].

The relationship between supersymmetry and topological invariants such as the indices of the elliptic operators is our main motivation for seeking general quantum mechanical symmetries with topological properties similar to those of supersymmetry.

Various generalizations of supersymmetry have been considered in the literature [7, 8, 9, 10, 11, 12]. Among these only extended and generalized supersymmetries [12] and a certain class of $p = 2$ parasupersymmetries [13] are known to share the topological characteristics of supersymmetry.

The strategy pursued in this article is as follows. First, we introduce the notion of a topological symmetry (TS) by formulating a set of basic principles that ensure the desired topological properties. Then, we investigate the underlying algebraic structure of these symmetries. This is necessary for seeking a mathematical interpretation of the corresponding topological invariants.

We have recently reported our preliminary results on $\mathbb{Z}_2$-graded TSs of type $(1,1)$ and $(2,1)$ in [14]. The purpose of the present article is to generalize the results of [14] to arbitrary $\mathbb{Z}_n$-graded TSs.

The organization of the article is as follows. In section 2, we give the definition of a general $\mathbb{Z}_n$-graded TS and introduce the associated topological invariants. In section 3, we consider the case of $n = 2$ and derive the algebra of a $\mathbb{Z}_2$-graded TS of arbitrary type (m_+, m_-). In particular, we show that for $m_- = 1$, the algebra can be reduced to the algebra of supersymmetry or $p = 2$ parasupersymmetry. In section 4, we consider the $\mathbb{Z}_n$-graded TSs. Here we discuss the properties of the grading operator and derive the algebra of $\mathbb{Z}_n$-graded TSs of arbitrary type $(m_1, m_2, \cdots, m_n)$. In section 5, we comment on the mathematical interpretation of the topological invariants associated with TSs. In section 6, we study some concrete examples of quantum systems possessing TSs. In

section 7, we summarize our results and present our concluding remarks. The appendix includes the proof of some of the mathematical results that we use in our analysis.

2 $\mathbb{Z}_n$-Graded TSs

In order to describe the concept of a topological symmetry, we first give some basic definitions. In the following we shall only consider the quantum systems with a self-adjoint Hamiltonian H. We shall further assume that all the energy levels are at most finitely degenerate.

Definition 1: Let n be an integer greater than 1, $\mathcal{H}$ denote the Hilbert space of a quantum system, and $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_n$ be (nontrivial) subspaces of $\mathcal{H}$. Then a state vector is said to have *definite color* c_ℓ iff it belongs to $\mathcal{H}_\ell$.

Definition 2: A quantum system is said to be *$\mathbb{Z}_n$-graded* iff its Hilbert space is the direct sum of n of its (nontrivial) subspaces $\mathcal{H}_\ell$, and its Hamiltonian has a complete set of eigenvectors with definite color.

Definition 3: Let m_ℓ be positive integers for all $\ell \in \{1, 2, \cdots, n\}$, and $m := \sum_{\ell=1}^{n} m_\ell$. Then a quantum system is said to possess a *$\mathbb{Z}_n$-graded topological symmetry of type* $(m_1, m_2, \cdots, m_n)$ iff the following conditions are satisfied.

a) The quantum system is $\mathbb{Z}_n$-graded;

b) The energy spectrum is nonnegative;

c) For every positive energy eigenvalue E, there is a positive integer λ_E such that E is $\lambda_E m$-fold degenerate, and the corresponding eigenspaces are spanned by $\lambda_E m_1$ vectors of color c_1, $\lambda_E m_2$ vectors of color c_2, $\cdots$, and $\lambda_E m_n$ vectors of color c_n.

Definition 4: A topological symmetry is said to be *uniform* iff for all positive energy eigenvalues E, $\lambda_E = 1$.

Theorem 1: Consider a quantum system possessing a $\mathbb{Z}_n$-graded topological symmetry of type $(m_1, m_2, \cdots, m_n)$, and let $n_\ell^{(0)}$ denote the number of zero-energy states of color c_ℓ. Then for all $i, j \in \{1, 2, \cdots, n\}$, the integers

$$\Delta_{ij} := m_i n_j^{(0)} - m_j n_i^{(0)} \tag{1}$$

remain invariant under continuous symmetry-preserving changes of the quantum system.[1]

Proof: The proof of this theorem is essentially the same as the proof of the topological invariance of the Witten index of supersymmetry. A symmetry-preserving continuous change of the quantum system will preserve the particular degeneracy and grading structures of positive energy levels. Because under such a change an initial zero energy eigenstate can only become a positive energy eigenstate and vice versa, the only possible change in the number of zero energy eigenstates are the ones involving the changes of $n_i^{(0)}$ of the form

$$n_i^{(0)} \to \tilde{n}_i^{(0)} := n_i^{(0)} + k\ m_i\ , \tag{2}$$

where k is an integer greater than or equal to $-n_i^{(0)}/m_i$. Moreover, such a change must occur simultaneously for all $n_i^{(0)}$'s, i.e., the transformation (2) is valid for all $i \in \{1, 2, \cdots, n\}$. Therefore under such a symmetry-preserving change of the system,

$$\Delta_{ij} \to \tilde{\Delta}_{ij} := m_i \tilde{n}_j^{(0)} - m_j \tilde{n}_i^{(0)} = m_i(n_j^{(0)} + k m_j) - m_j(n_i^{(0)} + k m_i) = m_i n_j^{(0)} - m_j n_i^{(0)} = \Delta_{ij}\ ,$$

i.e., Δ_{ij}'s remain invariant. $\square$

A direct consequence of Theorem 1 is that (the value of) any function of Δ_{ij}'s is a topological invariant of the system. In particular, Δ_{ij}'s are the basic topological invariants. A typical example of a derived invariant is

$$\Delta := \frac{1}{2} \sum_{i,j=1}^{n} (\Delta_{ij})^2\ . \tag{3}$$

[1] Here we consider quantum systems whose energy spectrum depends on a set of continuous parameters. These parameters may be identified with coupling constants or geometric quantities entering the definition of the Hamiltonian and/or the Hilbert space. A continuous change of the system corresponds to a continuous change of these parameters.

Note that Δ is a measure of the existence of the zero-energy states.

Let us next observe that the topological symmetries are simple generalizations of supersymmetry. First we recall that the Hilbert space of a supersymmetric system is $\mathbb{Z}_2$-graded. We can relate the $\mathbb{Z}_2$-grading of the Hilbert space $\mathcal{H}$ to the existence of a 'parity' or 'grading' operator $\tau : \mathcal{H} \to \mathcal{H}$ satisfying

$$\tau^2 = 1 , \tag{4}$$

$$\tau^\dagger = \tau , \tag{5}$$

$$[H, \tau] = 0 . \tag{6}$$

Here we use a $\dagger$ to denote the adjoint of the corresponding operator. We can identify the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with the eigenspaces of τ,

$$\mathcal{H}_1 = \mathcal{H}_+ , \quad \mathcal{H}_2 = \mathcal{H}_- , \quad \mathcal{H}_\pm := \{\psi \in \mathcal{H} \mid \tau\psi = \pm\psi\} . \tag{7}$$

Now, consider the superalgebra

$$[H, \mathcal{Q}] = 0 , \tag{8}$$

$$\frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^\dagger\} = H , \tag{9}$$

$$\mathcal{Q}^2 = 0 , \tag{10}$$

of supersymmetric quantum mechanics with one nonself-adjoint symmetry generator $\mathcal{Q}$ satisfying

$$\{\mathcal{Q}, \tau\} = 0 . \tag{11}$$

It is well-known that using the superalgebra (8) – (10) together with the properties of the grading operator (4) – (6) and the symmetry generator (11), one can show that the conditions a) – c) of Definition 3, with $n = 2$ and $m_1 = m_2 = 1$, are satisfied. Therefore, supersymmetry is a $\mathbb{Z}_2$-graded TS of type $(1, 1)$. For a $\mathbb{Z}_2$-graded TS of type $(1, 1)$, there is a single basic topological invariant namely Δ_{11}. This is precisely the Witten index.

3 Algebraic Structure of $\mathbb{Z}_2$-Graded TSs

In this section we shall explore the algebraic structure of the $\mathbb{Z}_2$-graded TSs that fulfil the following conditions.

- The $\mathbb{Z}_2$-grading is achieved by a grading operator τ satisfying (4) – (6);
- There is a single nonself-adjoint symmetry generator $\mathcal{Q}$;
- $\mathcal{Q}$ is an odd operator, i.e., it satisfies Eq. (11).

We shall only treat the case of the uniform TSs. The algebraic structure of nonuniform TSs is easily obtained from that of the uniform topological symmetries (UTSs). In fact, the algebraic relations defining uniform and nonuniform TSs of the same type are identical.

In order to obtain the algebraic structures that support TSs, we shall use the information on the degeneracy structure of the corresponding systems and the properties of the grading operator and the symmetry generator to construct matrix representations of the relevant operators in the energy eigenspaces $\mathcal{H}_E$ with positive eigenvalue E. We shall use the notation O^E for the restriction of an operator O onto the eigenspace $\mathcal{H}_E$. Throughout this article E stands for a positive energy eigenvalue. The zero-energy eigenspace (kernel of H) is denoted by $\mathcal{H}_0$.

In view of Eqs. (6) and (8), τ^E and $\mathcal{Q}^E$ are $m \times m$ matrices acting in $\mathcal{H}_E$. We also have the trivial identity: $H^E = E\, I_m$, where I_m denotes the $m \times m$ unit matrix.

Next, we introduce the self-adjoint symmetry generators

$$Q_1 := \frac{1}{\sqrt{2}}\,(\mathcal{Q} + \mathcal{Q}^\dagger) \quad \text{and} \quad Q_2 := \frac{-i}{\sqrt{2}}\,(\mathcal{Q} - \mathcal{Q}^\dagger) \tag{12}$$

where $i := \sqrt{-1}$. Note that because τ is self-adjoint, we have

$$\{Q_j, \tau\} = 0 \quad \text{for} \quad j \in \{1, 2\}. \tag{13}$$

Now in view of Eq. (6), we can choose a basis in $\mathcal{H}_E$ in which τ is diagonal. Then using Eqs. (4), (12), (13), and the self-adjointness of Q_j, we obtain the following matrix

representations for τ^E, Q_j^E, and $\mathcal{Q}^E$.

$$\begin{aligned}
\tau^E &= \mathrm{diag}(\underbrace{1,1,\cdots,1}_{m_+\text{times}},\underbrace{-1,-1,\cdots,-1}_{m_-\text{times}})\,, &(14)\\
Q_j^E &= \begin{pmatrix} 0 & A_j \\ A_j^\dagger & 0 \end{pmatrix}\,, &(15)\\
\mathcal{Q}^E &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & A_1+iA_2 \\ A_1^\dagger+iA_2^\dagger & 0 \end{pmatrix}\,, &(16)
\end{aligned}$$

where 'diag$(\cdots)$' stands for a diagonal matrix with diagonal entries '$\cdots$', 0's denote appropriate zero matrices, and A_j are $m_+\times m_-$ complex matrices.

The next step is to find general identities satisfied by Q_j^E and $\mathcal{Q}^E$ for all $E>0$. In order to derive the simplest such identities we appeal to the Cayley-Hamilton theorem of linear algebra. This theorem states that an $m\times m$ matrix Q satisfies its characteristic equations, $\mathcal{P}_Q(Q)=0$, where $\mathcal{P}_Q(x):=\det(xI_m-Q)$ is the characteristic polynomial for Q. Using this theorem we can prove the following lemma. The proof is given in the appendix.

Lemma 1: Let $m_\pm$ be positive integers, $m:=m_++m_-$, and Q be an $m\times m$ matrix of the form:

$$Q=\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}\,, \qquad (17)$$

where X and Y are $m_+\times m_-$ and $m_-\times m_+$ complex matrices. Let $\mathcal{P}_{XY}(x)$ and $\mathcal{P}_{YX}(x)$ denote the characteristic polynomials for XY and YX, respectively. Then $\mathcal{P}_{YX}(Q^2)Q=\mathcal{P}_{XY}(Q^2)Q=0$. Furthermore, if $m_+=m_-$, then $\mathcal{P}_{YX}(Q^2)=\mathcal{P}_{XY}(Q^2)=0$.

Applying this lemma to Q_j^E and $\mathcal{Q}^E$, we find for $m_+=m_-$

$$\begin{aligned}
\mathcal{P}_j[(Q_j^E)^2] &= 0\,, &(18)\\
\mathcal{P}[(\mathcal{Q}^E)^2] &= 0\,, &(19)
\end{aligned}$$

and for $m_+>m_-$

$$\begin{aligned}
\mathcal{P}_j[(Q_j^E)^2]Q_j^E &= 0\,, &(20)\\
\mathcal{P}[(\mathcal{Q}^E)^2]\mathcal{Q}^E &= 0\,, &(21)
\end{aligned}$$

where $\mathcal{P}_j(x)$ and $\mathcal{P}(x)$ denote the characteristic polynomials of $A_j^\dagger A_j$ and $(A_1^\dagger + iA_2^\dagger)(A_1 + iA_2)/2$, respectively,

For $m_- > m_+$, the roles of A_j and $A_j^\dagger$ are interchanged. We shall, therefore, restrict our attention to the case where $m_+ \geq m_-$.

We can write Eqs. (18) – (21) in terms of the roots of the characteristic polynomials $\mathcal{P}_j(x)$ and $\mathcal{P}(x)$. This yields

$$\begin{aligned} \left[(Q_j^E)^2 - \mu_{j1}^E I_m\right]\left[(Q_j^E)^2 - \mu_{j2}^E I_m\right]\cdots\left[(Q_j^E)^2 - \mu_{jm_-}^E I_m\right](Q_j^E)^{1-\delta(m_+,m_-)} &= 0\ , \quad (22)\\ \left[(\mathcal{Q}^E)^2 - \kappa_1^E I_m\right]\left[(\mathcal{Q}^E)^2 - \kappa_2^E I_m\right]\cdots\left[(\mathcal{Q}^E)^2 - \kappa_{m_-}^E I_m\right](\mathcal{Q}^E)^{1-\delta(m_+,m_-)} &= 0\ , \quad (23) \end{aligned}$$

where $\mu_{j\ell}^E$ and κ_ℓ^E are the roots[2] of $\mathcal{P}_j(x)$ and $\mathcal{P}(x)$, respectively, and

$$\delta(m_+,m_-) = \delta_{m_+,m_-} := \begin{cases} 1 & \text{for} \quad m_+ = m_- \\ 0 & \text{for} \quad m_+ \neq m_-\ . \end{cases}$$

In order to promote Eqs. (22) and (23) to operator relations, we introduce the operators $M_{j\ell}$ and $\mathcal{K}_\ell$ (for each $j \in \{1,2\}$ and $\ell \in \{1,2,\cdots,m_-\}$) which commute with H and have the representations:

$$M_{j\ell}^E = \mu_{j\ell}^E I_m \quad \text{and} \quad \mathcal{K}_\ell^E = \kappa_\ell^E I_m \tag{24}$$

in $\mathcal{H}_E$. We then deduce from Eqs. (22), (23) and (24) that $M_{j\ell}$ and $\mathcal{K}_\ell$ must commute with τ and Q_j. Furthermore, they should satisfy the algebra

$$\begin{aligned} (Q_j^2 - M_{j1})(Q_j^2 - M_{j2})\cdots(Q_j^2 - M_{jm_-})Q_j^{1-\delta(m_+,m_-)} &= 0\ , \quad (25)\\ (\mathcal{Q}^2 - \mathcal{K}_1)(\mathcal{Q}^2 - \mathcal{K}_2)\cdots(\mathcal{Q}^2 - \mathcal{K}_{m_-})\mathcal{Q}^{1-\delta(m_+,m_-)} &= 0\ . \quad (26) \end{aligned}$$

Note that the roots κ_ℓ^E and $\mu_{j\ell}^E$ are defined using the matrices A_j. This suggests that the operators $M_{j\ell}$ and $\mathcal{K}_\ell$ are not generally independent. Furthermore, because $A_j^\dagger A_j$ are Hermitian matrices, the roots $\mu_{j\ell}^E$, which are in fact the eigenvalues of $A_j^\dagger A_j$, are real. This in turn suggests that the operators $M_{j\ell}$ are self-adjoint.

In summary, the algebra of general $\mathbb{Z}_2$-graded topological symmetry of type (m_+,m_-) which is generated by one odd nonself-adjoint generator $\mathcal{Q}$ is given by Eqs. (25) and (26)

[2] Note that the roots are not necessarily distinct.

where m_+ is assumed (without loss of generality) not to be smaller than m_-, the operators $M_{j\ell}$ and $\mathcal{K}_\ell$ commute with H, τ and $\mathcal{Q}$, and $M_{j\ell}$ are self-adjoint. Moreover, $M_{j\ell}$ and $\mathcal{K}_\ell$ have similar degeneracy structure as the Hamiltonian[3] (at least for positive energy eigenvalues). In particular, it might be possible to express H as a function of $M_{j\ell}$ and $\mathcal{K}_\ell$.

In order to elucidate the role of the operators $M_{j\ell}$ and $\mathcal{K}_\ell$ and their relation to the Hamiltonian, we shall next consider the $\mathbb{Z}_2$-graded UTSs of type $(m_+, 1)$.

If $m_- = 1$, then $A_j^\dagger A_j$ and $(A_1^\dagger + iA_2^\dagger)(A_1 + iA_2)/2$ are respectively real and complex scalars. In this case, $\mathcal{P}_j(x) = x - \mu_j^E$ and $\mathcal{P}(x) = x - \kappa^E$, where

$$\mu_j^E = A_j^\dagger A_j \ , \quad \kappa^E = \frac{1}{2}\,(A_1^\dagger + iA_2^\dagger)(A_1 + iA_2) = \frac{1}{2}\,[(A_1^\dagger A_1 - A_2^\dagger A_2) + i(A_1^\dagger A_2 + A_2^\dagger A_1)] \ , \tag{27}$$

and the algebra (25) and (26) takes the form

$$(Q_j^2 - M_j)Q_j^{1-\delta(m_+,1)} = 0 \ , \tag{28}$$

$$(\mathcal{Q}^2 - \mathcal{K})\mathcal{Q}^{1-\delta(m_+,1)} = 0 \ . \tag{29}$$

Here we have used the abbreviated notation: $M_j = M_{j1}$ and $\mathcal{K} = \mathcal{K}_1$.

Next, we define the self-adjoint operators

$$K_1 = \mathcal{K} + \mathcal{K}^\dagger \quad \text{and} \quad K_2 = -i(\mathcal{K} - \mathcal{K}^\dagger) \ . \tag{30}$$

In view of Eqs. (24) and (27) we have

$$M_2 = M_1 - K_1 \ . \tag{31}$$

In the following we shall consider the cases $m_+ = 1$ and $m_+ > 1$ separately.

3.1 $\mathbb{Z}_2$-Graded UTS of Type $(1,1)$

Setting $m_+ = 1$ in Eqs. (28) and (29), we find

$$Q_j^2 = M_j \ , \tag{32}$$

$$\mathcal{Q}^2 = \mathcal{K} \ . \tag{33}$$

[3]The degeneracy structure of these operators will be the same as that of the Hamiltonian, if their eigenvalues $\mu_{j\ell}^E$ and κ_ℓ^E are distinct for different E.

If we express $\mathcal{Q}$ in terms of Q_j and use Eqs. (30) and (31), we can write Eqs. (32) and (33) in the form

$$
\begin{aligned}
Q_1^2 &= M_1 \ , & (34)\\
Q_2^2 &= M_1 - K_1 \ , & (35)\\
\{Q_1, Q_2\} &= K_2 \ . & (36)
\end{aligned}
$$

Now, we observe that Eqs. (34) – (36) remain form-invariant under the linear transformations of the form

$$
\begin{aligned}
Q_1 \to \tilde{Q}_1 = a\ Q_1 + b\ Q_2 \ ,\\
Q_2 \to \tilde{Q}_2 = c\ Q_1 + d\ Q_2 \ ,
\end{aligned} \tag{37}
$$

where $a,\ b,\ c$ and d are self-adjoint operators commuting with all other operators. More specifically, $\tilde{Q}_j$ satisfy Eqs. (34) – (36) provided that M_1 and K_j are transformed according to

$$
\begin{aligned}
M_1 &\longrightarrow \tilde{M}_1 := (a^2 + b^2)M_1 - b^2 K_1 + abK_2 \ , & (38)\\
K_1 &\longrightarrow \tilde{K}_1 := (a^2 + b^2 - c^2 - d^2)M_1 + +(d^2 - b^2)K_1 + (ab - cd)K_2 \ , & (39)\\
K_2 &\longrightarrow \tilde{K}_2 := 2(ac + bd)M_1 - 2bdK_1 + (ad + bc)K_2 \ . & (40)
\end{aligned}
$$

In particular, there are transformations of the form (37) for which $\tilde{K}_j = 0$. These correspond to the choices for $a,\ b,\ c$ and d that satisfy (either of)

$$
\frac{a + ic}{b + id} = -\frac{K_2}{2M_1} \pm i\sqrt{1 - \frac{K_1}{M_1} - \frac{K_2^2}{4M_1^2}} \ . \tag{41}
$$

One can use the representations of K_j and M_1 in the eigenspaces $\mathcal{H}_E$ to show that the terms in the square root in (41) yield a positive self-adjoint operator, provided that the kernel of M_1 is a subspace of the zero-energy eigenspace $\mathcal{H}_0$.

The above analysis shows that we can reduce the general algebra (34) – (36) to the special case where $K_j = 0$. Writing this algebra in terms of $\mathcal{Q}$, we obtain the superalgebra (8) – (10) with M_1 replacing H. In other words, if we identify the Hamiltonian with M_1, which we can always do, the algebra of $\mathbb{Z}_2$-graded topological symmetry of type $(1,1)$ reduces to that of supersymmetry.

3.2 $\mathbb{Z}_2$-Graded UTSs of Type $(m_+, 1)$ with $m+ > 1$

If $M_+ > 1$, then Eqs. (28) and (29) take the form

$$Q_j^3 = M_j Q_j \ , \tag{42}$$

$$\mathcal{Q}^3 = \mathcal{K}\mathcal{Q} \ . \tag{43}$$

Again we express $\mathcal{Q}$ in terms of Q_j and use Eqs. (30) and (31) to write (42) and (43) in the form

$$Q_1^3 = M_1 Q_1 \ , \tag{44}$$

$$Q_2^3 = (M_1 - K_1) Q_2 \ , \tag{45}$$

$$Q_2 Q_1 Q_2 + \{Q_1, Q_2^2\} = (M_1 - K_1) Q_1 + K_2 Q_2 \ , \tag{46}$$

$$Q_1 Q_2 Q_1 + \{Q_2, Q_1^2\} = M_1 Q_2 + K_2 Q_1 \ . \tag{47}$$

It is remarkable that these relations are also invariant under the transformations (37) and (38) – (40). Therefore, again we can reduce our analysis to the special case where $K_j = 0$. Substituting zero for K_j in Eqs. (44) – (47), and writing them in terms of $\mathcal{Q}$, we obtain

$$[M_1, \mathcal{Q}] = 0 \ , \tag{48}$$

$$\{\mathcal{Q}^2 Q^\dagger\} + \mathcal{Q}\mathcal{Q}^\dagger \mathcal{Q} = 2M_1 \mathcal{Q} \ , \tag{49}$$

$$\mathcal{Q}^3 = 0 \ . \tag{50}$$

This is precisely the algebra of $p = 2$ parasupersymmetry of Rubakov and Spiridonov [7] with H replaced by $M_1/2$. Hence, if we identify H with $M_1/2$, which we can always do, the algebra of $\mathbb{Z}_2$-graded topological symmetry of type $(m_+, 1)$ with $m_+ > 1$ reduces to that of the $p = 2$ parasupersymmetry.

As shown in Ref. [15], one can use the algebra (48) – (50) of $p = 2$ parasupersymmetry and properties of the grading operator (4) – (6) and (13) to obtain the general degeneracy structure of a $p = 2$ parasupersymmetric system. In general the algebra of $p = 2$ parasupersymmetry does not imply the particular degeneracy structure of the $\mathbb{Z}_2$-graded UTS of type $(m_+, 1)$, even for $m_+ = 2$. Therefore, the $\mathbb{Z}_2$-graded UTS of type $(2, 1)$ is

a subclass of the general $p = 2$ parasupersymmetries. As argued in Refs. [15] and [13], these are parasupersymmetries for which an analog of the Witten index can be defined.

In Ref. [15] it is also shown that the positive energy eigenvalues of a $p = 2$ parasupersymmetric system can at most be triply degenerate, provided that the eigenvalues of Q_1^E for all $E > 0$ are nondegenerate. This means that the $\mathbb{Z}_2$-graded TSs of type $(m_+, 1)$ with $m_+ > 2$ occur only if Q_1^E have degenerate eigenvalues for all $E > 0$. This suggests the presence of further (even) symmetry generators L_a which would commute with Q_1 and label the basis eigenvectors within the degeneracy subspaces of Q_1. The existence of these generators is an indication that the $\mathbb{Z}_2$-graded TSs of type $(m_+, 1)$ with $m_+ > 2$ are not uniform.

3.3 Special $\mathbb{Z}_2$-Graded TSs

The analysis of the $\mathbb{Z}_2$-graded TSs of type $(m_+, 1)$ shows that the corresponding algebras can be reduced to a simplified special case by a redefinition of the symmetry generators Q_j. This raises the question whether this is also possible for the general $\mathbb{Z}_2$-graded TSs of type (m_+, m_-). The reduction made in the case of $\mathbb{Z}_2$-graded TSs of type $(m_+, 1)$ has its roots in the form of the matrix representation of the corresponding symmetry generators in $\mathcal{H}_E$. For a $\mathbb{Z}_2$-graded UTS of arbitrary type (m_+, m_-), a similar reduction, which eliminates the operators $\mathcal{K}_\ell$ in Eq. (26), is possible, if we can find a transformation $Q_j \to \tilde{Q}_j$ which satisfies the following conditions.

1) The transformed generators $\tilde{Q}_j$ have the representation

$$\tilde{Q}_j^E = \begin{pmatrix} 0 & \tilde{A}_j \\ \tilde{A}_j^\dagger & 0 \end{pmatrix}$$

in $\mathcal{H}_E$.

2) For all energy eigenvalues $E > 0$, the corresponding matrices $\tilde{A}_j$ which define $\tilde{Q}_j^E$ satisfy $\tilde{A}_2 = U\tilde{A}_1$, where U is an $m_+ \times m_+$ unitary and anti-Hermitian matrix.

The first condition is necessary for the invariance of the algebra (25) – (26). It is satisfied by the linear transformations: $Q_j^E \to \tilde{Q}_j^E = T_j Q_j^E T_j^\dagger$ where T_j are $m \times m$ matrices

of the form

$$T_j = \begin{pmatrix} T_{j+} & 0 \\ 0 & T_{j-} \end{pmatrix} ,$$

and $T_{j\pm}$ are $m_\pm \times m_\pm$ matrices. Under such a transformation A_j transform according to $A_j \to \tilde{A}_j := T_{j+} A_j T_{j-}^\dagger$.

The second condition implies that the transformed matrices $\tilde{A}_j$ satisfy

$$(\tilde{A}_1^\dagger + i\tilde{A}_2^\dagger)(\tilde{A}_1 + i\tilde{A}_2) = A_1^\dagger (I_{m_+} + iU^\dagger)(I_{m_+} + iU) A_1 = A_1^\dagger [(I_{m_+} - U^\dagger U) + i(U + U^\dagger)] A_1 = 0 , \tag{51}$$

where we have used the unitarity and anti-Hermiticity of U. The latter relation indicates that for the transformed system $\kappa_\ell^E = 0$ and $\mu_{2\ell}^E = \mu_{1\ell}^E$, for all E and ℓ. Hence $\mathcal{K}_\ell$ can be identified with the zero operator and $M_{2\ell} = M_{1\ell} =: M_\ell$. Furthermore, one can check that (51) implies $(\tilde{\mathcal{Q}}^E)^{3-\delta(m_+,m_-)} = 0$. Therefore, the transformed generators satisfy

$$(\tilde{Q}_j^2 - M_1)(\tilde{Q}_j^2 - M_2) \cdots (\tilde{Q}_j^2 - M_{m_-}) \tilde{Q}_j^{1-\delta(m_+,m_-)} = 0 , \tag{52}$$

$$\tilde{\mathcal{Q}}^{3-\delta(m_+,m_-)} = 0 . \tag{53}$$

We shall term such $\mathbb{Z}_2$-graded TSs, the *special* $\mathbb{Z}_2$-graded TSs.

3.4 $\mathbb{Z}_2$-graded TSs with one Self-Adjoint Generator

The above analysis of $\mathbb{Z}_2$-graded TSs can be easily applied to the case where there is a single self-adjoint generator Q. In fact, one can read off the corresponding algebra from Eqs. (25) and (26). The result is

$$(Q^2 - M_1)(Q^2 - M_2) \cdots (Q^2 - M_{m_-}) Q^{1-\delta(m_+,m_-)} = 0 , \tag{54}$$

where M_ℓ are self-adjoint operators commuting with H, Q and τ. For $m_\pm = 1$, this equation reduces to that of the $N = \frac{1}{2}$ supersymmetry, provided that we identify M_1 with the Hamiltonian H.

4 Algebraic Structure of $\mathbb{Z}_n$-Graded TSs

In the preceding section, we have used Definition 3 and the properties (4) – (6) and (11) of the $\mathbb{Z}_2$-grading operator τ to obtain the algebra of the $\mathbb{Z}_2$-graded TSs with one

generator. The main guideline for postulating these properties were Definition 2 and the known grading structure of supersymmetric systems.

Similarly, in order to obtain the algebraic structure of the $\mathbb{Z}_n$-graded TSs, with $n > 2$, we must first postulate the existence of an appropriate $\mathbb{Z}_n$-grading operator. In view of Definitions 1, 2 and 3, such a grading operator τ must commute with the Hamiltonian — Eq. (6) holds — and have n-distinct eigenvalues c_ℓ with eigenspaces $\mathcal{H}_\ell$ satisfying $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \mathcal{H}_n$. The simplest choice for c_ℓ are the n-th roots of unity, e.g., $c_\ell = q^\ell$ where $q = e^{2\pi i/n}$. This choice suggests the following generalization of Eqs. (4) and (5).

$$\tau^n = 1 \tag{55}$$

$$\tau^\dagger = \tau^{-1} \tag{56}$$

Note that for $n = 2$, according to Eq. (4), $\tau^{-1} = \tau$ and Eq. (56) coincides with (5).

In the following we shall only consider $\mathbb{Z}_n$-graded TSs with one symmetry generator $\mathcal{Q}$. In order to proceed along the same lines as in the case of $n = 2$, we need to impose a grading condition on $\mathcal{Q}$ similar to (11). We first consider the simplest case, namely $\mathbb{Z}_n$-graded UTS of type $(1, 1, \cdots, 1)$.

4.1 $\mathbb{Z}_n$-Graded UTS of Type $(1, 1, \cdots, 1)$

For $n = 2$, condition (11) implies that the action of $\mathcal{Q}$ on a definite color (parity) state vector changes its color (parity) – a bosonic state changes to a fermionic state and vice versa. The simplest generalization of this statement to the case $n > 2$ is that the action of $\mathcal{Q}$ must change the color of a definite color state by one unit, i.e.,

$$\tau\psi = c_\ell\psi \quad \text{implies} \quad \tau(\mathcal{Q}\psi) = c_{\ell+1}Q\psi \ . \tag{57}$$

This condition is consistent with Eq. (55). If we impose this condition on the representations $\mathcal{Q}^E$ and τ^E in $\mathcal{H}_E$, then in a basis in which τ^E is diagonal we have

$$\tau^E = \mathrm{diag}(q, q^2, \cdots, q^{n-1}, q^n = 1) \ , \tag{58}$$

and

$$\mathcal{Q}^E = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & a_n \\ a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1} & 0 \end{pmatrix} , \tag{59}$$

where a_ℓ are complex numbers depending on E.

A simple calculation shows that τ^E and $\mathcal{Q}^E$ q-commute, i.e. $[\tau^E, \mathcal{Q}^E]_q = 0$, where the q-commutator is defined by $[O_1, O_2]_q := O_1O_2 - qO_2O_1$. Generalizing this property of τ^E and $\mathcal{Q}^E$ to τ and $\mathcal{Q}$, we find

$$[\tau, \mathcal{Q}]_q = 0 \; . \tag{60}$$

This relation is the algebraic expression of the condition (57). It reduces to Eq. (11) for $n = 2$.

Another consequence of Eq. (59) is that a symmetry generator of a $\mathbb{Z}_n$-graded UTS of type $(1, 1, \cdots, 1)$ with $n > 2$ which satisfies (57) cannot be self-adjoint. Furthermore, one can easily check that

$$(\mathcal{Q}^E)^n = a_1a_2 \cdots a_n I_{n\times n} \; . \tag{61}$$

We can generalize this equation to the whole Hilbert space and write it in the operator form:

$$\mathcal{Q}^n = \mathcal{K} \; . \tag{62}$$

Here $\mathcal{K}$ is an operator that commutes with all other operators in the algebra.

Next, we seek for the algebraic relations satisfied by the self-adjoint generators:

$$Q_1 := \frac{1}{\sqrt{2}}(\mathcal{Q} + \mathcal{Q}^\dagger) \quad \text{and} \quad Q_2 := \frac{-i}{\sqrt{2}}(\mathcal{Q} - \mathcal{Q}^\dagger) \; . \tag{63}$$

Using the results reported in the appendix, namely Eq. (141), one can show that in an energy eigenspace $\mathcal{H}_E$, with $E > 0$, Q_1 and Q_2 satisfy

$$(Q_1^E)^n + \alpha_{n-2}(Q_1^E)^{n-2} + \cdots \;\; = \;\; (\frac{1}{\sqrt{2}})^n R_1 \; , \tag{64}$$

$$(Q_2^E)^n + \alpha_{n-2}(Q_2^E)^{n-2} + \cdots \;\; = \;\; (\frac{1}{\sqrt{2}})^n R_2 \; , \tag{65}$$

where α_ℓ's are functions of $|a_i|^2$ and R_1 and R_2 are defined by

$$R_1 := \prod_{k=1}^{n} a_k + \prod_{k=1}^{n} a_k^* , \qquad R_2 := \prod_{k=1}^{n}(-ia_k) + \prod_{k=1}^{n}(ia_k^*) .$$

Eqs. (64) and (65) can be written in the operator form according to

$$Q_1^n + M_{n-2}Q_1^{n-2} + \cdots = (\frac{1}{\sqrt{2}})^n(\mathcal{K}+\mathcal{K}^\dagger) , \tag{66}$$

$$Q_2^n + M_{n-2}Q_2^{n-2} + \cdots = (\frac{1}{\sqrt{2}})^n(i^n\mathcal{K}^\dagger + (-i)^n\mathcal{K}) , \tag{67}$$

where M_is are self-adjoint operators commuting with all other operators.

We can rewrite Eqs. (66) and (67) in the following more symmetric way.

For $n = 2p$,

$$\begin{array}{c} (Q_1^2 - \mathcal{M}_1)(Q_1^2 - \mathcal{M}_2)\cdots(Q_1^2 - \mathcal{M}_p) = (\frac{1}{2})^p(\mathcal{K}+\mathcal{K}^\dagger) , \\ (Q_2^2 - \mathcal{M}_1)(Q_2^2 - \mathcal{M}_2)\cdots(Q_2^2 - \mathcal{M}_p) = (\frac{-1}{2})^p(\mathcal{K}+\mathcal{K}^\dagger) . \end{array} \tag{68}$$

For $n = 2p+1$,

$$\begin{array}{c} (Q_1^2 - \mathcal{M}_1)(Q_1^2 - \mathcal{M}_2)\cdots(Q_1^2 - \mathcal{M}_p)Q_1 = (\frac{1}{\sqrt{2}})^{2p+1}(\mathcal{K}+\mathcal{K}^\dagger) , \\ (Q_2^2 - \mathcal{M}_1)(Q_2^2 - \mathcal{M}_2)\cdots(Q_2^2 - \mathcal{M}_p)Q_2 = (\frac{i}{\sqrt{2}})^{2p+1}(\mathcal{K}^\dagger-\mathcal{K}) . \end{array} \tag{69}$$

Here $\mathcal{M}_i$ are also operators that commute with all other operators. Note also that for even p's, the algebraic relations for Q_1 and Q_2 coincide.

Eqs. (62), (68), and (69) are the defining equations of the algebra of $\mathbb{Z}_n$-graded TS of type $(1,1,\cdots,1)$. For $n = 2$, they reduce to the familiar algebra of $\mathbb{Z}_2$-graded TS of type $(1,1)$. In this case, as we discussed in section 3.1, we can perform a linear transformation on the self-adjoint generators that eliminates the operator $\mathcal{K}$.

Next, consider the case where for all $E > 0$, $\mathcal{K}^E \neq 0$ and introduce the transformed generator $\tilde{\mathcal{Q}}$ by

$$\tilde{\mathcal{Q}}^E = \left(\frac{E}{|\kappa^E|}\right)^{1/n} e^{-i\phi^E/n}\mathcal{Q}^E , \tag{70}$$

where

$$\kappa^E := a_1a_2\cdots a_n \quad \text{and} \quad e^{i\phi^E} := \frac{\kappa^E}{|\kappa^E|} . \tag{71}$$

In view of Eqs. (61) and (70), it is not difficult to see that

$$(\tilde{\mathcal{Q}}^E)^n = EI_n , \tag{72}$$

Writing this equation in the operator form, we are led to

$$H = \tilde{\mathcal{Q}}^n \ . \tag{73}$$

This is precisely the algebra of fractional supersymmetry of order n [10, 11].

Next, we examine whether Eq. (62) guarantees the desired degeneracy structure of the $\mathbb{Z}_n$-graded TS of type $(1, 1, \cdots, 1)$. In order to address this question, we suppose that the kernel of $\mathcal{K}$ is a subset of $\mathcal{H}_0$. Then for all $E > 0$, $\kappa^E \neq 0$.

In view of Eq. (62), the eigenvalues of $\mathcal{Q}^E$ are of the form $q^\ell(\kappa^E)^{1/n}$, with $\ell \in \{1, 2, \cdots, n\}$. Now, let $|q^\ell, \nu_\ell\rangle$ denote the corresponding eigenvectors, where ν_ℓ's are degeneracy labels. We can easily show using Eq. (60) that for all positive integers s, $\tau^s|q^\ell, \nu_\ell\rangle$ are eigenvectors of $\mathcal{Q}^E$ with eigenvalue $q^{\ell-s}(\kappa^E)^{1/n}$. This is sufficient to conclude that all the eigenvalues of $\mathcal{Q}^E$ are either nondegenerate or have the same multiplicity N_E. If for all $E > 0$, $N_E = 1$, then we will have the desired degeneracy structure of the $\mathbb{Z}_n$-graded UTS of type $(1, 1, \cdots, 1)$. If there are $E > 0$ for which $N_E > 1$, then we have a nonuniform $\mathbb{Z}_n$-graded TS of type $(1, 1, \cdots, 1)$.

We conclude this section by noting that if we put $\mathcal{K} = 0$ in the algebraic relations for $\mathbb{Z}_n$-graded UTS of type $(1, 1, \cdots, 1)$, we will obtain the algebraic relations for $\mathbb{Z}_2$-graded TS. This is not so strange, because putting $\mathcal{K} = 0$ means that one of the a_i's (say a_n) in $\mathcal{Q}^E$ is zero. In this case, as we explain in the appendix, there is a unitary transformation that transforms Q_1^E and Q_2^E to off block diagonal matrices. In view of the analysis of section 3, this implies that the generators of this kind of $\mathbb{Z}_n$-graded UTSs satisfy the algebra of $\mathbb{Z}_2$ graded UT.

4.2 $\mathbb{Z}_n$-Graded TS of Arbitrary Type $(m_1, m_2, \cdots, m_n)$

In order to obtain the algebraic structure of the $\mathbb{Z}_n$-graded TS of arbitrary type $(m_1, m_2, \cdots, m_n)$, we need an appropriate grading operator. We shall adopt Eqs. (6), (55), (56) and (60) as the defining conditions for our $\mathbb{Z}_n$-grading operator. We shall further assume that $m_1 \leq m_2 \leq \cdots \leq m_n$. This ordering can always be achieved by a reassignment of the colors. Again we shall confine our attention to the uniform $\mathbb{Z}_n$-graded TS. The algebras

of uniform and nonuniform TS of the same type are identical.

Working in an eigenbasis in which τ^E is diagonal and using Eq. (60), we have

$$\tau^E = \mathrm{diag}(\underbrace{q, q, \cdots, q}_{m_1 \text{times}}, \underbrace{q^2, q^2, \cdots, q^2}_{m_2 \text{times}}, \cdots, \underbrace{q^n, q^n, \cdots, q^n}_{m_n \text{times}}) \,. \tag{74}$$

In view of this equation and Eq. (60), we obtain the following matrix representation for $\mathcal{Q}^E$.

$$\mathcal{Q}^E = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & A_n \\ A_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & A_{n-1} & 0 \end{pmatrix} , \tag{75}$$

where A_ℓ (with $\ell \in \{1, 2, \cdots, n-1\}$) are complex $m_{\ell+1} \times m_\ell$ matrices, A_n is a complex $m_1 \times m_n$ matrix, and 0's are appropriate zero matrices.

Next, we compute the n-th power of $\mathcal{Q}$. The result is

$$(\mathcal{Q}^E)^n = \begin{pmatrix} A_n A_{n-1} \cdots A_2 A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_1 A_n A_{n-1} \cdots A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{n-1} A_{n-2} \cdots A_1 A_n \end{pmatrix} . \tag{76}$$

In order to find the most general algebraic identity satisfied by $\mathcal{Q}$ we appeal to the following generalization of Lemma 1. The proof is given in the appendix.

Lemma 2: Let $(m_1, m_2, \cdots, m_n)$ be an n-tuple of positive integers satisfying $m_1 \leq m_2 \leq \cdots m_n$, $m := \sum_{\ell=1}^n m_\ell$, δ is the number of times m_1 appears in $(m_1, m_2, \cdots, m_n)$, Q is an $m \times m$ matrix of the form (75), and $\mathcal{P}(x)$ is the characteristic polynomial of the $m_1 \times m_1$ matrix $A_n A_{n-1} \cdots A_2 A_1$. Then Q satisfies

$$\mathcal{P}(Q^n) Q^{n-\delta} = 0 \,. \tag{77}$$

Substituting $\mathcal{Q}^E$ for Q in Eq. (77) and writing $\mathcal{P}(x)$ in terms of its roots κ_k^E, we find

$$\left[(\mathcal{Q}^E)^n - \kappa_1^E\right] \left[(\mathcal{Q}^E)^n - \kappa_2^E\right] \cdots \left[(\mathcal{Q}^E)^n - \kappa_{m_1}^E\right] (\mathcal{Q}^E)^{n-\delta} = 0 \,. \tag{78}$$

Next, we introduce the operators $\mathcal{K}_k$ for $k \in \{1, 2, \cdots, m_1\}$ which commute with the Hamiltonian and have the representation:

$$\mathcal{K}_k^E = \kappa_k^E \; I_m \tag{79}$$

in $\mathcal{H}_E$. In view of Eqs. (78) and (79), we obtain

$$(\mathcal{Q}^n - \mathcal{K}_1)(\mathcal{Q}^n - \mathcal{K}_2) \cdots (\mathcal{Q}^n - \mathcal{K}_{m_1})\mathcal{Q}^{n-\delta} = 0 \; . \tag{80}$$

The operators $\mathcal{K}_k$ have a similar degeneracy structure as the Hamiltonian (at least for the positive energy eigenvalues). Therefore, the Hamiltonian might be expressed as a function of $\mathcal{K}_k$.

For a $\mathbb{Z}_n$-graded UTS of type $(1, 1, \cdots, 1)$, $\delta = n$ and Eq. (80) reduces to (62). However, one can show that in general Eq. (80) does not ensure the desired degeneracy structure of the general $\mathbb{Z}_n$-graded TSs. This is true even for the case $n = 2$, $m_+ = 2$, $m_- = 1$ considered in section 3. In general, the $\mathbb{Z}_n$-graded TSs correspond to a special class of symmetries satisfying (80).

Finally, we wish to note that for a general $\mathbb{Z}_n$-graded TS the algebraic relations satisfied by the self-adjoint generators are extremely complicated. We have not been able to express them in a closed form.

5 Mathematical Interpretation of $\Delta_{i,j}$

In order to obtain the mathematical interpretation of the invariants $\Delta_{i,j}$ of TSs, one must express the Hamiltonian in terms of the symmetry generators. This can be easily done using the defining algebra for the $\mathbb{Z}_n$-graded TSs of type $(1, 1, \cdots, 1)$. In the following, we discuss the mathematical meaning of the topological invariants associated with these symmetries.

We know that for a $\mathbb{Z}_n$-graded TS of type $(1, 1, \cdots, 1)$ the operator $\mathcal{K}$ has a similar degeneracy structure as the Hamiltonian. Therefore, we may set $H = f(\mathcal{K})$ where f is a function mapping the eigenvalues κ^E to E. Next, suppose that the kernels of $\mathcal{K}$ and H also coincide. Then as far as the general properties of the symmetry is concerned, we can

confine our attention to the special case where $\mathcal{K} = H$, i.e., the fractional supersymmetry. Note that in this case, $\mathcal{Q}^n$ is necessarily self-adjoint. Alternatively, we can use the rescaled symmetry generator $\tilde{\mathcal{Q}}$ that does satisfy $\tilde{\mathcal{Q}}^n = H$.

In order to obtain the mathematical interpretation of the topological invariants $\Delta_{i,j}$, we use an n-component representation of the Hilbert space in which a state vector ψ is represented by a column of n colored vectors $\psi_\ell \in \mathcal{H}_\ell$. In this representation the grading operator is diagonal and the generator of the symmetry and the Hamiltonian are respectively expressed by $n \times n$ matrices of operators according to

$$\mathcal{Q} = \left(\begin{array}{ccccccc} 0 & 0 & 0 & \cdots & 0 & 0 & D_n \\ D_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & D_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \cdots & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & D_{n-1} & 0 \end{array} \right), \tag{81}$$

$$H = \mathcal{Q}^n = \left(\begin{array}{cccccc} H_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & H_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & H_n \end{array} \right). \tag{82}$$

Here $D_n : \mathcal{H}_n \to \mathcal{H}_1$ and $D_\ell : \mathcal{H}_\ell \to \mathcal{H}_{\ell+1}$, with $\ell \in \{1, 2, \cdots, n-1\}$, are operators and

$$\begin{aligned} H_1 &:= D_n D_{n-1} \cdots D_2 D_1 \,, \\ H_2 &:= D_1 D_n D_{n-1} \cdots D_2 \,, \\ \cdots & \quad \cdots \\ H_n &:= D_{n-1} D_{n-2} \cdots D_1 D_n \,. \end{aligned}$$

The condition that H is self-adjoint takes the form $H_\ell^\dagger = H_\ell$, or alternatively

$$D_{\sigma(n)} D_{\sigma(n-1)} \cdots D_{\sigma(2)} D_{\sigma(1)} = D_{\sigma(1)}^\dagger D_{\sigma(2)}^\dagger \cdots D_{\sigma(n-1)}^\dagger D_{\sigma(n)}^\dagger \,, \tag{83}$$

for all cyclic permutations σ of $(n, n-1, n-2, \cdots 1)$. In addition, the assumption that H has a nonnegative spectrum further restricts D_ℓ. Note also that the algebraic relations satisfied by the self-adjoint generators also put restrictions on the choice of the operators

D_ℓ. This is because the operators M_i appearing in Eqs. (66) and (67) involve D_ℓ. The condition that M_i commute with $\mathcal{Q}$ leads to a set of compatibility relations among D_ℓ.

In view of Eq. (82) and the fact that $n_\ell^{(0)}$ is the dimension of the kernel of H_ℓ, we can easily express the invariant Δ_{ij} in the form:

$$\Delta_{i,j} = \dim(\ker\, H_j) - \dim(\ker\, H_i)\ . \tag{84}$$

Suppose that the subspaces $\mathcal{H}_\ell$ are all identified with a fixed Hilbert Space. Consider the special case where

$$D_3 = D_4 = \cdots = D_n = 1, \qquad D_2 = D_1^\dagger, \tag{85}$$

and D_1 is a Fredholm operator, then $H_1 = D_1^\dagger D_1$, $H_2 = D_1 D_1^\dagger$, and there is one independent invariant, namely

$$\Delta_{1,2} = \dim(\ker\, D_1^\dagger D_1) - \dim(\ker\, D_1 D_1^\dagger) = \dim(\ker\, D_1) - \dim(\ker\, D_1^\dagger)\ .$$

This is just the analytic index of D_1. This example shows that the above construction has nontrivial solutions.

In general, the operator D_ℓ need not satisfy (85). They are however subject to the above-mentioned compatibility relations. In order to demonstrate the nature of these relations, we consider $\mathbb{Z}_3$-graded UTS of type $(1,1,1)$ with the symmetry generator

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 & D_3 \\ D_1 & 0 & 0 \\ 0 & D_2 & 0 \end{pmatrix}. \tag{86}$$

In this case, the algebraic relations (66) and (67) satisfied by the self-adjoint generators involve a single commuting operator which we denote by M. Assuming that this operator has the form

$$M = \frac{1}{2}\begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix} \tag{87}$$

and enforcing

$$[\mathcal{Q}, M] = 0, \quad Q_1^3 + MQ_1 = \frac{1}{\sqrt{2}} H\ , \tag{88}$$

we find

$$M_1 D_3 = D_3 M_3 \tag{89}$$

$$M_2 D_1 = D_1 M_1 \tag{90}$$

$$M_3 D_2 = D_2 M_2 \tag{91}$$

One can manipulate these relations to obtain the following compatibility relations for D_ℓ.

$$D_2^\dagger D_2 D_1 D_3 = D_1 D_3 D_2 D_2^\dagger \,, \tag{92}$$

$$D_3^\dagger D_3 D_2 D_1 = D_2 D_1 D_3 D_3^\dagger \,, \tag{93}$$

$$D_1^\dagger D_1 D_3 D_2 = D_3 D_2 D_1 D_1^\dagger \,. \tag{94}$$

Furthermore, under these conditions on M_ℓ and D_ℓ, one can check that the relation for Q_2, i.e., $Q_2^3 + M Q_2 = 0$, is identically satisfied.

Next, consider the case where one of the D_i's, say D_3, is 1. Then we can use Eqs. (89) – (91) to express M_i in terms of D_1 and D_2. This yields

$$M_1 = M_3 = -(D_1^\dagger D_1 + D_2 D_2^\dagger + 1)\,, \quad M_2 = -(D_2^\dagger D_2 + D_1 D_1^\dagger + 1)\,. \tag{95}$$

Note that we can easily satisfy Eqs. (92) – (94), if $D_2 = D_1^\dagger$, i.e., when (85) is satisfied. In this case,

$$M_1 = M_2 = -(2D_1^\dagger D_1 + 1) = -(2H_1 + 1)\,, \quad M_3 = -(2D_1 D_1^\dagger + 1) = -(2H_2 + 1)\,, \tag{96}$$

and

$$M = -(H + \frac{1}{2})\,, \tag{97}$$

where we have used Eqs. (82) and (87).

6 Examples

In this section, we examine some examples of quantum systems possessing UTSs.

6.1 A System with UTS of Type $(2,1)$

Consider the Hamiltonian

$$H = \frac{1}{2}(p^2 + x^2) + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{98}$$

where x and p are respectively the position and momentum operators. This Hamiltonian was originally considered in [8]. It is not difficult to show that it commutes with

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}}(p - ix) \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}(p + ix) & 0 \end{pmatrix}. \tag{99}$$

Furthermore, $\mathcal{Q}$, H, and the the self-adjoint generators Q_j satisfy the algebra of the UTS of type $(2,1)$, i.e.,

$$Q_1^3 = HQ_1\,, \qquad Q_2^3 = HQ_2\,, \qquad \mathcal{Q}^3 = 0\,.$$

For this system the grading operator is given by $\tau = \mathrm{diag}(1,1,-1)$; there is a nondegenerate zero-energy ground state; and the positive energy eigenvalues are triply degenerate. Therefore, this system has a UTS of type $(2,1)$.

If we denote by $|n\rangle$ the normalized energy eigenvectors of the harmonic oscillator with unit mass and frequency, then a set of eigenvectors of the Hamiltonian (98) are given by

$$|\phi_0\rangle = \begin{pmatrix} 0 \\ 0 \\ |0\rangle \end{pmatrix}, \tag{100}$$

for $E = 0$, and

$$|\phi_n, 1\rangle = \begin{pmatrix} |n-1\rangle \\ 0 \\ 0 \end{pmatrix},\ |\phi_n, 2\rangle = \begin{pmatrix} 0 \\ |n-1\rangle \\ 0 \end{pmatrix},\ |\phi_n, 3\rangle = \begin{pmatrix} 0 \\ 0 \\ |n\rangle \end{pmatrix}, \tag{101}$$

for $E = n > 0$.

Next, we check the action of the symmetry generator and the grading operator on $|\phi_0\rangle$ and $|\phi_n, a\rangle$. This yields

$$\mathcal{Q}|\phi_0\rangle = 0, \qquad \tau|\phi_0\rangle = -|\phi_0\rangle, \tag{102}$$

$$\mathcal{Q}|\phi_n,1\rangle = 0, \quad \mathcal{Q}|\phi_n,2\rangle \to |\phi_n,3\rangle, \quad \mathcal{Q}|\phi_n,3\rangle \to |\phi_n,1\rangle\,, \tag{103}$$

$$\tau|\phi_n,1\rangle = |\phi_n,1\rangle, \quad \tau|\phi_n,2\rangle = |\phi_n,2\rangle, \quad \tau|\phi_n,3\rangle = -|\phi_n,3\rangle. \tag{104}$$

Here we have used $\to$ to denote equality up to a nonzero multiplicative constant.

As one can see from Eqs. (102) and (104), the ground state has negative parity and the positive energy levels consist of two positive and one negative parity states. The topological invariants for this system are $\Delta_{2,1} = -\Delta_{1,2} = -2$.

6.2 A System with UTS of Type $(1,1,1)$

Consider the Hamiltonian

$$H = \mathcal{Q}^3 = \frac{1}{2}(p^2 + x^2) + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{105}$$

which is precisely the Hamiltonian (98) written in another basis.[4] It possesses a symmetry generated by

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}}(p+ix) & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}(p-ix) & 0 \end{pmatrix}. \tag{106}$$

In fact, it is not difficult to show that

$$\mathcal{Q}^3 = H\,. \tag{107}$$

Furthermore, one can check that the self-adjoint generators Q_j satisfy

$$Q_1^3 + MQ_1 = \frac{1}{\sqrt{2}}\,H\,, \quad Q_2^3 + MQ_2 = 0\,, \tag{108}$$

where the operator M is given by

$$M = -\frac{1}{2}\,(p^2 + x^2) + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -(H + \frac{1}{2})\,. \tag{109}$$

This is in agreement with the more general treatment of section 5. In particular, Eq. (109) is a special case of Eq. (97).

[4]We have used some of the results of [11] to obtain this Hamiltonian.

Obviously, M commutes with H and $\mathcal{Q}$. In view of this observation and Eqs. (107), (108), (62), (66), and (67) we can conclude that this system has a $\mathbb{Z}_3$-graded UTS of type $(1,1,1)$.

A complete set of eigenvectors of the Hamiltonian (98) are given by

$$|\phi_0\rangle = \begin{pmatrix} 0 \\ |0\rangle \\ 0 \end{pmatrix}, \tag{110}$$

for $E = 0$,and

$$|\phi_n, 1\rangle = \begin{pmatrix} |n-1\rangle \\ 0 \\ 0 \end{pmatrix}, \ |\phi_n, 2\rangle = \begin{pmatrix} 0 \\ |n\rangle \\ 0 \end{pmatrix}, \ |\phi_n, 3\rangle = \begin{pmatrix} 0 \\ 0 \\ |n-1\rangle \end{pmatrix}, \tag{111}$$

for $E = n > 0$.

The grading operator is $\tau = \mathrm{diag}(q, q^2, 1)$, where $q := e^{2\pi i/3}$. The symmetry generator $\mathcal{Q}$ and the grading operator τ transform the energy eigenvectors according to

$$\mathcal{Q}|\phi_0\rangle = 0\,, \qquad \tau|\phi_0\rangle = q^2|\phi_0\rangle\,, \tag{112}$$

$$\mathcal{Q}|\phi_n, 1\rangle \to |\phi_n, 2\rangle, \qquad \mathcal{Q}|\phi_n, 2\rangle \to |\phi_n, 3\rangle, \quad \mathcal{Q}|\phi_n, 3\rangle \to |\phi_n, 1\rangle. \tag{113}$$

$$\tau|\phi_n, 1\rangle = q|\phi_n, 1\rangle, \qquad \tau|\phi_n, 2\rangle = q^2|\phi_n, 2\rangle, \quad \tau|\phi_n, 3\rangle = |\phi_n, 3\rangle. \tag{114}$$

In particular, $|\phi_0\rangle$ and $|\phi_n, a\rangle$ have colors q^2 and q^a, respectively, and the topological invariants of the system are given by

$$\Delta_{1,2} = -\Delta_{2,1} = 1, \quad \Delta_{2,3} = -\Delta_{3,2} = -1, \quad \Delta_{1,3} = -\Delta_{3,1} = 0. \tag{115}$$

6.3 A system with UTS of type$(1, 1, \cdots, 1)$

Consider the Hamiltonian

$$H = \frac{1}{2}(p^2 + x^2) + \frac{1}{2}\,\mathrm{diag}(\underbrace{1, 1, \cdots, 1}_{\text{n}-1\text{times}}, -1)\,. \tag{116}$$

This Hamiltonian has a symmetry generated by

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2}}(p-ix) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{2}}(p+ix) & 0 \end{pmatrix} , \tag{117}$$

because $\mathcal{Q}^n = H$.

One can also check that the self-adjoint generators Q_j satisfy

$$Q_1^n + M_{n-2} Q_1^{n-2} + \cdots \;=\; (\frac{1}{\sqrt{2}})^n (2H) \,, \tag{118}$$

$$Q_2^n + M_{n-2} Q_2^{n-2} + \cdots \;=\; (\frac{i}{\sqrt{2}})^n (1 + (-1)^n) H \,, \tag{119}$$

where M_{n-2k} are given by

$$M_{n-2k} = (-1)^k \left[\frac{1}{2^k} \binom{n-k-1}{k} + \frac{1}{2^{k-1}} \binom{n-k-1}{k-1} H \right] , \tag{120}$$

and $\begin{pmatrix} a \\ b \end{pmatrix} := \frac{a!}{b!(a-b)!}$.

It is not difficult to see that this system has a zero-energy ground state and that the positive energy levels are n-fold degenerate. A complete set of energy eigenvectors are given by

$$|0\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ |0\rangle \end{pmatrix} , \tag{121}$$

for $E = 0$, and

$$|m,1\rangle = \begin{pmatrix} |m-1\rangle \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} , \cdots , |m,n-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ |m-1\rangle \\ 0 \end{pmatrix} , \quad |m,n\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ |m\rangle \end{pmatrix} , \tag{122}$$

for $E = m > 0$:

These observations indicate that the quantum system defined by the Hamiltonian (116) has a $\mathbb{Z}_n$-graded UTS of type $(1,1,\cdots,1)$. Clearly, the grading operator is $\tau = \mathrm{diag}(q, q^2, \cdots, q^{n-1}, q^n = 1)$, the ground state has color $c_n = q^n = 1$, and the nonvanishing topological invariants are $\Delta_{\ell,n} = -\Delta_{n,\ell} = 1$, where $\ell \in \{1, 2, \cdots, n-1\}$.

Next, we wish to comment that if we change the sign of the term involving the matrix $\mathrm{diag}(\underbrace{1,1,\cdots,1}_{\mathrm{n-1times}},-1)$ in the Hamiltonian (116), then we obtain another quantum system with a $\mathbb{Z}_n$-graded UTS of type $(1,1,\cdots,1)$ that is generated by

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2}}(p+ix) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{2}}(p-ix) & 0 \end{pmatrix}, \qquad (123)$$

This system has an $(n-1)$-fold degenerate zero-energy ground state.

7 Conclusions

We have introduced a generalization of supersymmetry that shares its topological properties. We gave a complete description of the underlying algebraic structure and commented on the meaning of the corresponding topological invariants.

We showed that the algebras of the $\mathbb{Z}_2$-graded TSs of type $(m_+, 1)$ coincide with the algebras of supersymmetry or parasupersymmetry of order $p = 2$. The algebraic relations obtained for the $\mathbb{Z}_2$-graded TSs of type (m_+, m_-) with $m_- > 1$ include as special cases the algebras of higher order parasupersymmetry advocated by Durand and Vinet [16]. We also pointed out that the algebra of $\mathbb{Z}_n$-graded TS of type $(1,1,\cdots,1)$ is related to the algebra of fractional supersymmetry of order n.

Our approach in developing the concept of a TS differs from those of the other generalizations of supersymmetry in the sense that we introduce TSs in terms of certain requirements on the spectral degeneracy properties of the corresponding quantum systems, whereas in the other generalizations of supersymmetry such as parasupersymmetry and fractional supersymmetry one starts with certain defining algebraic relations. These

relations are usually obtained by generalizing the relations satisfied by the generators of symmetries that relate degrees of freedom with different statistical properties in certain simple models. For example the Robakov-Spiridonov algebra of parasupersymmetry of order $p = 2$ was originally obtained by generalizing the algebra of symmetry generators of an oscillator involving a bosonic and a ($p = 2$) parafermionic degree of freedom [7]. In order to investigate the topological content of these (statistical) generalizations of supersymmetry, one is forced to study the spectral degeneracy structure of the corresponding systems. The derivation of the degeneracy structure using the defining algebraic relations is usually a difficult task. In fact, for parasupersymmetries of order $p > 2$ this problem has not yet been solved. Even for the parasupersymmetries of order $p = 2$ the solution requires a quite lengthy analysis [15], and the defining algebra does not guarantee the existence of any topological invariants. This in turn raises the question of the classification of the $p = 2$ parasupersymmetries that do have topological properties similar to supersymmetry [13]. The analysis of the topological aspects of supersymmetry and $p = 2$ parasupersymmetry shows that the information about the topological properties is contained in the spectral degeneracy structure of the corresponding systems. This is the main justification for our definition of a TS.

Like any other quantum mechanical symmetry, a TS also possesses an underlying operator algebra. This algebra contains more practical information about the systems possessing the symmetry. As we showed in the preceding sections, the operator algebras associated with TSs can be obtained using the defining conditions on the spectral degeneracy structure of these systems. This observation may be viewed as another indication that, as far as the topological aspects are concerned, the spectral degeneracy structure is more basic than the algebraic structure.

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Appendix

In this appendix we give the proofs of some of the mathematical results we use in sections 3 and 4. In the following we shall denote the characteristic polynomial of a matrix M by $\mathcal{P}_M(x)$, i.e., $\mathcal{P}_M(x) = \det(xI - M)$.

Lemma 0: Let X and Y be $m \times n$ and $n \times m$ matrices respectively. Then

$$\mathcal{P}_{YX}(\lambda) = \lambda^{n-m}\mathcal{P}_{XY}(\lambda)\,. \tag{124}$$

Proof: Let

$$M := \begin{pmatrix} I_m & X \\ Y & \lambda I_n \end{pmatrix}. \tag{125}$$

Then using the well-known properties of the determinant, we have

$$\begin{aligned} \det(M) &= \det\begin{pmatrix} I_m & X \\ Y & \lambda I_n \end{pmatrix} \\ &= \det\begin{pmatrix} I_m & X \\ 0 & \lambda I_n - YX \end{pmatrix} \\ &= det(\lambda I_n - YX) \\ &= \mathcal{P}_{YX}(\lambda)\,. \end{aligned} \tag{126}$$

Similarly, we can show that

$$\begin{aligned} \det(M) &= \det\begin{pmatrix} I_m & X \\ Y & \lambda I_n \end{pmatrix} \\ &= \lambda^n \det\begin{pmatrix} I_m & X \\ \frac{1}{\lambda}Y & I_n \end{pmatrix} \\ &= \lambda^{n-m} \det\begin{pmatrix} \lambda I_m & \lambda X \\ \frac{1}{\lambda}Y & I_n \end{pmatrix} \\ &= \lambda^{n-m} det\begin{pmatrix} \lambda I_m - XY & 0 \\ \frac{1}{\lambda}Y & I_n \end{pmatrix} \\ &= \lambda^{n-m} \det(\lambda I_m - XY) \\ &= \lambda^{n-m}\mathcal{P}_{XY}(\lambda)\,. \end{aligned} \tag{127}$$

Eqs. (126) and (127) yield the identity (124). □

Lemma 1: Let $m_\pm$ be positive integers, $m = m_+ + m_-$, and Q be an $m \times m$ matrix of the form

$$Q = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}, \tag{128}$$

where X and Y are $m_+ \times m_-$ and $m_- \times m_+$ complex matrices. Then $\mathcal{P}_{XY}(Q^2)Q = \mathcal{P}_{YX}(Q^2)Q = 0$. Furthermore, if $m_+ = m_-$, then $\mathcal{P}_{XY}(Q^2) = \mathcal{P}_{YX}(Q^2) = 0$.

Proof: Let a_k denote the coefficients of $\mathcal{P}_{YX}(x)$, i.e., $\mathcal{P}_{YX}(x) = \sum_{k=0}^n a_k x^k$. According to the Cayley-Hamilton theorem,

$$\mathcal{P}_{YX}(YX) = \sum_{k=0}^n a_k (YX)^k = 0. \tag{129}$$

In view of this identity, we can easily show that for any positive integer k,

$$Q^{2k} = \begin{pmatrix} (XY)^k & 0 \\ 0 & (YX)^k \end{pmatrix}, \tag{130}$$

$$\begin{aligned} \mathcal{P}_{YX}(Q^2) &= \sum_{k=0}^n a_k Q^{2k} = \begin{pmatrix} \sum_{k=0}^n a_k (XY)^k & 0 \\ 0 & \sum_{k=0}^n a_k (YX)^k \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^n a_k (XY)^k & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{131}$$

$$\begin{aligned} \mathcal{P}_{YX}(Q^2)Q &= \begin{pmatrix} 0 & \sum_{k=0}^n a_k (XY)^k X \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & X\sum_{k=0}^n a_k (YX)^k \\ 0 & 0 \end{pmatrix} = 0. \end{aligned} \tag{132}$$

A similar calculation yields $\mathcal{P}_{XY}(Q^2)Q = 0$. Furthermore, using Lemma 0 one can see that if $m_+ = m_-$, then $\mathcal{P}_{XY}(x) = \mathcal{P}_{YX}(x)$. In view of this identity and Eq. (131), we have (for the case $m_+ = m_-$) $\mathcal{P}_{XY}(Q^2) = \mathcal{P}_{YX}(Q^2) = 0$. □

Corollary 1: Consider the $n \times n$ matrix M whose elements are given by

$$M_{ij} := a_j \delta_{i,j+1} + a_i^* \delta_{i+1,j}\ , \quad i, j \in \{1, 2, \cdots, n\} \tag{133}$$

i.e.,

$$M := \begin{pmatrix} 0 & a_1^* & 0 & \cdots & 0 & 0 \\ a_1 & 0 & a_2^* & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1}^* \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \end{pmatrix} \tag{134}$$

Then the characteristic polynomial of M is given by

$$\mathcal{P}_M(x) = \begin{cases} \mathcal{P}_{AA^\dagger}(x^2) & \text{for } n = 2p \\ \mathcal{P}_{AA^\dagger}(x^2)x & \text{for } n = 2p+1, \end{cases} \tag{135}$$

where A is the matrix with entries

$$A_{ij} = a_{2i-1}\delta_{ij} + a_{2i}^*\delta_{i+1,j} \ . \tag{136}$$

It is a $p \times p$ matrix for $n = 2p$ and a $p \times (p+1)$ matrix for $n = 2p+1$.

Proof: Consider the following unitary transformation

$$M \to \tilde{M} = U^\dagger M U \ , \tag{137}$$

where U is defined by

$$U_{ij} = \begin{cases} \delta_{i,2j} & \text{for } j \leq p \\ \delta_{i,2j-2p-1} & \text{for } j > p. \end{cases} \tag{138}$$

Substituting this equation in (137), we find

$$\tilde{M} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \ . \tag{139}$$

Now applying Lemma 1, we obtain Eq. (135). Furthermore, the coefficeints of the characteristic polynomial of M are functions of $|a_i|^2$s only. $\square$

Corollary 2: Consider the $n \times n$ complex matrix

$$Q := \begin{pmatrix} 0 & a_1^* & 0 & \cdots & 0 & a_n \\ a_1 & 0 & a_2^* & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1}^* \\ a_n^* & 0 & 0 & \cdots & a_{n-1} & 0 \end{pmatrix} \ . \tag{140}$$

Then the characteristic polynomial of Q is given by

$$\mathcal{P}_Q(x) = -(\prod_{k=1}^{n} a_k + \prod_{k=1}^{n} a_k^*) + x^n + \beta_{n-2}x^{n-2} + \cdots + \beta_{n-2k}x^{n-2k} + \cdots \qquad (141)$$

where β_{n-2k}s are functions of $|a_i|^2$s.

Proof: A straightforward application of the properties of the determinant, one can show that

$$\mathcal{P}_Q(x) = -(\prod_{k=1}^{n} a_k + \prod_{k=1}^{n} a_k^*) + \mathcal{P}_M(x) - |a_n|^2\mathcal{P}'_M(x) \;, \qquad (142)$$

where M' is obtained from M by removing the first and last rows and columns. Now, since $\mathcal{P}_M(x)$ is an odd (even) polynomial for odd (even) n, $\mathcal{P}_Q(x)$ will have the form given by Eq. (141). □

Lemma 2: Let $(m_1, m_2, \cdots, m_n)$ be an n-tuple of positive integers satisfying $m_1 \leq m_2 \leq \cdots m_n$, $m := \sum_{\ell=1}^{n} m_\ell$, δ is the number of times m_1 appears in $(m_1, m_2, \cdots, m_n)$, Q is an $m \times m$ matrix of the form (75), and $\mathcal{P}(x)$ is the characteristic polynomial of the $m_1 \times m_1$ matrix $A_n A_{n-1} \cdots A_2 A_1$. Then Q satisfies

$$\mathcal{P}(Q^n)Q^{n-\delta} = 0 \,. \qquad (143)$$

Sketch of Proof: The proof of this lemma is very similar to the proof of Lemma 1. The idea is to multiply the block diagonal matrix $\mathcal{P}(Q^n)$ with a power of Q so that one obtains a matrix whose entries have a factor $\mathcal{P}(A_n A_{n-1} \cdots A_1)$. Then one uses the Cayley-Hamilton theorem to conclude that the resulting matrix must vanish. One can show by inspection that the smallest nonnegative integer r for which a factor of $\mathcal{P}(A_n A_{n-1} \cdots A_1)$ occurs in all the entries of $\mathcal{P}(Q^n)Q^r$ is $n - \delta$.

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